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## Hidden local, quasi-local and non-local Symmetries in Integrable Systems

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### Abstract

The knowledge of *non usual* and sometimes *hidden* symmetries of (classical) integrable systems provides a very powerful setting-out of solutions of these models. Primarily, the understanding and possibly the quantisation of intriguing symmetries could give rise to deeper insight into the nature of field spectrum and correlation functions in quantum integrable models. With this perspective in mind we will propose a general framework for discovery and investigation of local, quasi-local and non-local symmetries in classical integrable systems. We will pay particular attention to the structure of symmetry algebra and to the rôle of conserved quantities. We will also stress a nice unifying point of view about KdV hierarchies and Toda field theories with the result of obtaining a Virasoro algebra as exact symmetry of Sine-Gordon Model.

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# 1 Introduction.

What is usually defined as a 1+1 dimensional Integrable System is a classical or quantum field theory with the property to have an infinite number of *local integrals of motion in involution* (LIMI), among which the hamiltonian (energy) operator . This kind of symmetry does not allow the determination of the most intriguing and interesting features of a system because of its abelian character. Instead, the presence of an infinite dimensional non-abelian algebra could *complete* the abelian algebra giving rise to the possibility of building its representations, *i.e.* the spectrum (of the energy) and the spectrum of fields. We may name this non-commuting algebra a spectrum generating algebra. In different models and in a mysterious way the presence of this spectrum generating symmetry is very often connected to the abelian one. This is the case of the *simplest* integrable quantum theories – the two Dimensional Conformal Field Theories (**2D-CFT's**) – their common crucial property being covariance under the infinite dimensional Virasoro symmetry, a true spectrum generating symmetry. Indeed, the Verma modules (highest weight representations) of this algebra classify all the local fields in 2D-CFT's and turn out to be reducible because of the occurrence of vectors of null hermitian product with all other vectors, the so called *null-vectors*. The factorization by the modules generated over the null-vectors leads to a number of very interesting algebraic-geometrical properties such as fusion algebras, differential equations for correlation functions, etc. .

Unfortunately this beautiful picture collapses when one pushes the system away from criticality by perturbing the original CFT with some relevant local field  $\Phi$ :

$$S = S_{CFT} + \mu \int d^2z \Phi(z, \bar{z}), \quad (1.1)$$

and from the infinite dimensional Virasoro symmetry only the Poincaré subalgebra survives the perturbation. Consequently, one of the most important open problems in two Dimensional Quantum Field Theories (**2D-IQFT's**) is the construction of the spectrum of the local fields and consequently the *computation* of their correlation functions. Actually, the CFT possesses a bigger  $\mathcal{W}$ -like symmetry and in particular it is invariant under an infinite dimensional abelian subalgebra of the latter [1]. With suitable deformations, this abelian subalgebra survives the perturbation (1.1), resulting in the so-called LIMI. As said before, this symmetry does not carry sufficient information, and in particular one cannot build the spectrum of an integrable theory of type (1.1) by means of LIMI alone.

In the literature, there are several attempts to find spectrum generating algebras at least at conformal point. For instance in [2], some progress was made in arranging the spectrum generated through the action of objects called *spinons* in representations of deformed algebras. The kind of structure built by spinons eliminates automatically all null vectors and consequently is unclear how to get (equations for) correlation functions. Besides, the extension of this method in the scaling limit out of criticality is up to now

unknown. More recently, it has been conjectured in [3] that one could add to LIMI  $I_{2m+1}$  additional non-commuting *charges*  $J_{2m}$  in such a way that the resulting *algebra* (actually it is not clear from [3] if these objects close an algebra) would be sufficient to create all the states of a particular class of perturbed theory (Restricted Sine-Gordon Theory) of type (1.1). Therein it was also discovered that a sort of null-vector condition appears in the above procedure leading to certain equations for the form factors. However, what remains unclear in [3] are the field theory expressions of null vectors, the general procedure for finding them and above all the symmetry structures lying behind their arising. Besides, heavy use is made of the very specific form of the form factors of the RSG model, and it is not clear how to extend this procedure to other integrable theories of the form (1.1). What is promising in this work is the link the authors created between (quantum) Form Factors and classical solutions of the equation of motion.

With this connection in mind, we present here a general framework to investigate symmetries and related charges in 2D integrable classical field theories. The plan of the paper is as follows.

In Section 2 we present the central idea, *i.e.* basing the whole construction on a generalization of the so-called Dressing Symmetry Transformations [4] connecting these to the usual way of finding integrable systems [5]. In fact, our basic objects will be the *transfer matrix*  $T(x; \lambda)$ , which generates the dressing, and the *resolvent*  $Z(x; \lambda)$ , the dressed generator of the underlying symmetry. Although it is clear from the construction that our method is applicable to any generalized KdV hierarchy [5], we will be concerned with the semiclassical limit [6, 7, 8] of minimal CFT's [9], namely the  $A_1^{(1)}$ -KdV and the  $A_2^{(2)}$ -KdV systems [5]. Besides, we will show how to obtain in a geometrical way from these conformal hierarchies the non conformal Sine-Gordon Model (SGM), *i.e.* the semiclassical limit of Perturbed Conformal Field Theories (PCFT's) (1.1). This kind of geometrical point of view on the SGM (Toda field theories, in general), is very useful in derivation of new symmetries and more general Toda field theories are linked to generalized KdV equations.

In Section 3 we will present an alternative approach to the description of the spectrum of the local fields in the classical limit of the 2D Integrable Field Theories. We will build a systematic and geometric method for deriving constraints or classical null vectors without the use of Virasoro algebra.

In Section 4 we will propose a further generalization of the aforementioned dressing transformations. The central idea is that we may dress not only the generators of the underlying Kac-Moody algebra but also differential operators in the spectral parameter,  $\lambda^m \partial_\lambda^n$ , forming a  $w_\infty$  algebra. The corresponding vector fields close a  $w_\infty$  algebra as well with a Virasoro subalgebra (realized for  $n = 1$ ) made up of quasi-local and non-local transformations. The regular non-local ones are expressed in terms of vertex operators and complete the quasi-local asymptotic ones to the full Virasoro algebra. All these vector fields do not commute with the KdV hierarchy flows, but have a sort of spectrum generating action on them. Besides, it is very intriguing that only the positive index

ones are exact symmetries of the SGM (apparently the negative index ones do not matter particularly in this theory). The apparition of a Virasoro symmetry, with its rich and well known structure, is particularly useful and interesting.

In Section 5 we will deal with the  $A_2^{(2)}$ -KdV showing, as an example of generalization, the building of the spectrum of local fields through the same geometrical lines as before.

In Section 6 we will summarize our results giving some hint on the meaning and quantisation of these symmetries in critical and off-critical theories.

## 2 The usual $A_1^{(1)}$ -mKdV.

### 2.1 Introductory remarks on integrability.

As already observed in [6], the classical limit ( $c \rightarrow -\infty$ ) of CFT's is described by the second Hamiltonian structure of the (usual) KdV which is built through the centerless Kac-Moody algebra  $A_1^{(1)}$  in the Drinfeld-Sokolov scheme [5]. In the classification [5] a generalized modified-KdV (mKdV) hierarchy is attached to each affine Kac-Moody loop algebra  $\mathcal{G}$ . Various Miura transformations [10] relate it to the generalized KdV hierarchies, each one classified by the choice of a node  $c_m$  of the Dynkin diagram of  $\mathcal{G}$ . However, nodes symmetrical under automorphisms of the Dynkin diagram lead to the same hierarchy. The classical Poisson structure of this hierarchy is a classical  $w(\tilde{\mathcal{G}})$ -algebra, where  $\tilde{\mathcal{G}}$  is the finite dimensional Lie algebra obtained by deleting the  $c_m$  node. In the simplest case the usual mKdV equation

$$\partial_t v = -\frac{3}{2}v^2 v' - \frac{1}{4}v''' \quad (2.1)$$

describes the temporal flow for the spatial derivative

$$v = -\phi' \quad (2.2)$$

of a Darboux field  $\phi$  defined on a spatial interval  $x \in [0, L]$ . Assuming quasi-periodic boundary conditions on  $\phi(x)$ , it verifies by definition the Poisson bracket

$$\{\phi(x), \phi(y)\} = \frac{1}{2}s(x-y), \quad (2.3)$$

with the quasi-periodic extension of the sign-function  $s(x)$  defined as

$$s(x) = 2n + 1, \quad nL < x < (n+1)L. \quad (2.4)$$

As all the generalized mKdV, the simplest one (2.1) can be re-written as a null curvature condition [11]

$$[\partial_t - A_t, \partial_x - A_x] = 0 \quad (2.5)$$

for connections belonging to the  $A_1^{(1)}$  loop algebra

$$\begin{aligned} A_x &= -vh + (e_0 + e_1), \\ A_t &= \lambda^2(e_0 + e_1 - vh) - \frac{1}{2}[(v^2 - v')e_0 + (v^2 + v')e_1] - \frac{1}{2}\left(\frac{v''}{2} - v^3\right)h \end{aligned} \quad (2.6)$$

where the generators  $e_0, e_1, h$  are chosen in the canonical gradation of the  $A_1^{(1)}$  loop algebra

$$e_0 = \lambda E \quad , \quad e_1 = \lambda F \quad , \quad h = H, \quad (2.7)$$

with  $E, F, H$  generators of  $A_1$  Lie algebra:

$$[H, E] = 2E \quad , \quad [H, F] = -2F \quad , \quad [E, F] = H. \quad (2.8)$$

For reasons of simplicity we will deal with the fundamental representation

$$e_0 = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \quad , \quad e_1 = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.9)$$

The KdV variable  $u(x, t)$  is related to the mKdV variable  $\phi(x)$  by the Miura transformation [10]

$$u(x) = -\phi'(x)^2 - \phi''(x), \quad (2.10)$$

which is the classical counterpart of the quantum Feigin-Fuchs transformation [12]. A remarkable geometrical interest is obviously attached to the transfer matrix which performs the parallel transport along the  $x$ -axis, *i.e.* the solution of the boundary value problem

$$\begin{aligned} \partial_x M(x; \lambda) &= A_x(x; \lambda) M(x; \lambda) \\ M(0; \lambda) &= \mathbf{1}. \end{aligned} \quad (2.11)$$

The formal solution of the previous equation

$$M(x, \lambda) = \mathcal{P}e^{\int_0^x A_x(y, \lambda) dy} \quad (2.12)$$

can be expressed by means of a series of non-negative powers of  $\lambda$  with an infinite convergence radius and non-local coefficients. The main property of the solution (2.12) is that it allows one to calculate the equal time Poisson brackets between the entries of the monodromy matrix

$$M(\lambda) = M(L; \lambda) = \mathcal{P}e^{\int_0^L A_x(y, \lambda) dy}, \quad (2.13)$$

provided those among the entries of the connection  $A_x$  are known [13]. The result of this calculation is that the Poisson brackets of the entries of the monodromy matrix are fixed by the so called classical  $r$ -matrix [13]

$$\{M(\lambda) \otimes M(\mu)\} = [r(\lambda\mu^{-1}), M(\lambda) \otimes M(\mu)]. \quad (2.14)$$

In our particular case the  $r$ -matrix is the trigonometric one (calculated, possibly, in the fundamental representation of  $sl(2)$ ):

$$r(\lambda) = \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}} \frac{H \otimes H}{2} + \frac{2}{\lambda - \lambda^{-1}} (E \otimes F + F \otimes E). \quad (2.15)$$

By carrying through the trace on both members of the Poisson brackets (2.14) we are allowed to conclude immediately that

$$\tau(\lambda) = tr M(\lambda) \quad (2.16)$$

Poisson-commute with itself for different values of the spectral parameter

$$\{\tau(\lambda), \tau(\mu)\} = 0. \quad (2.17)$$

In other words,  $\tau(\lambda)$  is the generating function of the conserved charges in involution, *i.e.* it guarantees the integrability of the model *à la* Liouville. Now, it is possible to expand the generating function  $\tau(\lambda)$  in two different independent ways in order to obtain two different sets of conserved charges in involution. One is the regular expansion in non-negative powers of  $\lambda$ , the other is the asymptotic expansion in negative powers. In the first case the coefficients in the Taylor series are non local charges of the theory, instead in the second one the coefficients in the asymptotic series are the LIM. Likewise the transfer matrix  $T(x, \lambda)$  can be expanded in the two ways just mentioned giving rise to different algebraic and geometric structures, as we will see in the following. The regular expansion is typically employed in the derivation of Poisson-Lie structures for Dressing Symmetries [4]. Instead, the second type of expansion plays a crucial rôle in obtaining the flows of the integrable hierarchy and the local integrals of motion that generate symplectically these flows [13, 5]. In this section we will see how the aforementioned approaches are actually reconducible to a single geometrical procedure which, moreover, produces two different kinds of symmetries.

## 2.2 Regular expansion of the transfer matrix and the Dressing Transformations.

The coefficients of the regular expansion of  $\tau$  (2.16) are non-local integrals of motion in involution (NLIM). However, these latter may be included in a larger non-abelian algebra of conserved charges, *i.e.* commuting with local hamiltonians of the mKdV (2.1), but not all between themselves. Actually to get those in a suitable form, we use a slightly different procedure than the usual one [4], considering a solution of the associated linear problem (2.11) with a different initial condition. Explicitly, we select the following solution of the first equation in (2.12) which contains *the fundamental primary field*  $e^\phi$  [14]:

$$T_{reg}(x; \lambda) = e^{H\phi(x)} \mathcal{P} \exp \left( \lambda \int_0^x dy (e^{-2\phi(y)} E + e^{2\phi(y)} F) \right) \quad (2.18)$$

or, equivalently, defining  $K(x) = e^{-2\phi(x)}E + e^{2\phi(x)}F$ ,

$$T_{reg}(x; \lambda) = e^{H\phi(x)} \sum_{k=0}^{\infty} \lambda^k \int_{x \geq x_1 \geq x_2 \geq \dots \geq x_k \geq 0} K(x_1)K(x_2)\dots K(x_k) dx_1 dx_2 \dots dx_k. \quad (2.19)$$

Now we apply the usual dressing techniques [4] using the previous expression  $T_{reg}(x; \lambda)$  and stressing the point of contact with the derivation of an integrable hierarchy given in [5]. If we define a *resolvent*  $Z^X(x, \lambda)$  for the Lax operator  $\mathcal{L} = \partial_x - A_x$  (2.6) as a solution of the equation

$$[\mathcal{L}, Z^X(x; \lambda)] = 0, \quad (2.20)$$

it turns out that we may build several solutions by mean of a dressing reformulation of the first equation of (2.11)

$$T\partial_x T^{-1} = \partial_x - A_x. \quad (2.21)$$

In the specific case of the regular expansion (2.19), if  $X = H, E, F$  is one of the generators (2.8) we get a regular *resolvent* by dressing

$$Z^X(x, \lambda) = (T_{reg} X T_{reg}^{-1})(x, \lambda) = \sum_{k=0}^{\infty} \lambda^k Z_k^X. \quad (2.22)$$

The definition (2.20) of the resolvent is the key property for the construction of a symmetry algebra, since, once the gauge connection

$$\Theta_n^X(x; \lambda) = (\lambda^{-n} Z^X(x; \lambda))_- = \sum_{k=0}^{n-1} \lambda^{k-n} Z_k^X \quad (2.23)$$

is constructed, the commutator  $[\mathcal{L}, \Theta_n^X(x; \lambda)]$  is in  $\lambda$  of the same degree of  $[\mathcal{L}, (\lambda^{-n} Z^X(x; \lambda))_+]$  and hence of degree zero. Therefore, to get a self-consistent gauge transformation

$$\Delta_n^X \mathcal{L} = [\Theta_n^X(x; \lambda), \mathcal{L}], \quad (2.24)$$

we have to require only that the r.h.s. in (2.24) is proportional to  $H$ . This depends, for  $X$  fixed, on whether  $n$  is even or odd. Indeed, a recursive relation between the terms  $Z_n^X$  in (2.22) follows straightforward from the definition (2.20)

$$\begin{aligned} \partial_x Z_0^X &= \phi'[H, Z_0^X] \\ \partial_x Z_n^X &= \phi'[H, Z_n^X] + [E + F, Z_{n-1}^X]. \end{aligned} \quad (2.25)$$

and allows us to find the modes  $Z_n^X$  once the different *initial conditions* are established by inserting the first term of the expansion (2.19) into (2.22)

$$Z_0^H = H \quad , \quad Z_0^E = e^{2\phi} E \quad , \quad Z_0^F = e^{-2\phi} F. \quad (2.26)$$

The previous two relations yield the various terms of the expansion of the resolvent in the form

$$\begin{aligned} Z_{2m}^H(x) &= a_{2m}^H(x)H \quad , \quad Z_{2m+1}^H(x) = b_{2m+1}^H(x)E + c_{2m+1}^H(x)F \\ Z_{2r}^E(x) &= b_{2r}^E(x)E + c_{2r}^E(x)F \quad , \quad Z_{2r+1}^E(x) = a_{2r+1}^E(x)H \\ Z_{2p}^F(x) &= b_{2p}^F(x)E + c_{2p}^F(x)F \quad , \quad Z_{2p+1}^F(x) = a_{2p+1}^F(x)H \end{aligned} \quad (2.27)$$

where  $a_n^X, b_n^X, c_n^X$  are non-local integral expressions, containing exponentials of the field  $\phi$ . In addition, the variation (2.24) may be explicitly calculated as

$$\Delta_n^X A_x = [Z_{n-1}^X, E + F] \quad (2.28)$$

and hence it is clear that  $Z_{n-1}^X$  cannot contain any term proportional to  $H$ . The conclusions about the parity of  $n$  of (2.28) are that

- in the  $Z^H$  case,  $n$  in (2.24) must be even,
- in the  $Z^E$  and  $Z^F$  case,  $n$  must conversely be odd.

From (2.25) it is possible to work out simple recursion relations for the coefficients  $a_n^X, b_n^X, c_n^X$  in (2.27)

$$\begin{aligned} a_n'^X &= c_{n-1}^X - b_{n-1}^X, \\ b_{n+1}^X - 2\phi' b_{n+1}^X + 2a_n^X &= 0, \\ c_{n+1}^X + 2\phi' c_{n+1}^X - 2a_n^X &= 0 \end{aligned} \quad (2.29)$$

where  $n$  is even for  $X = H$  and odd for  $X = E, F$ . In fact, another way to obtain this expansion could be to substitute directly the regular expansion (2.19) in (2.22), but the recursive relations (2.29) (with the initial conditions (2.26)) will provide a contact with the use of *the recursive operator* [15] in the theory of integrable hierarchies. Now, the action (2.28) of the symmetry generators on the bosonic field is given in terms of  $a_n^X(x)$  by making use of (2.29)

$$\Delta_n^X \phi' = -\partial_x a_n^X(x) \quad (2.30)$$

in which  $n$  is even for  $X = H$  and odd for  $X = E, F$ . Instead, using the previous equations of motions and the relations (2.29) it is simple to show that the action on the *classical stress-energy tensor*  $u(x)$  (2.10) is given in terms of  $b_{n-1}^X(x)$

$$\Delta_n^X u = -2\partial_x b_{n-1}^X(x), \quad (2.31)$$

in which  $n$  is even for  $X = H$  and odd for  $X = E, F$ . Now, we are interested in finding a recursive relation giving  $a_n^X$  in terms of  $a_{n+2}^X$  and vice versa. If we indicate with  $\partial_x^{-1} = \int_0^x dy$ , the system of equations (2.29) yields easily

$$\begin{aligned} a_n^X &= \mathcal{R}_a a_{n+2}^X \quad , \quad \mathcal{R}_a = -v\partial_x^{-1}v\partial_x + \frac{1}{4}\partial_x^2, \\ a_{n+2}^X &= \mathcal{R}_a^{-1} a_n^X \quad , \quad \mathcal{R}_a^{-1} = 2(\partial_x^{-1}e^{-2\phi}\partial_x^{-1}e^{2\phi} + \partial_x^{-1}e^{2\phi}\partial_x^{-1}e^{-2\phi}), \end{aligned} \quad (2.32)$$

in which  $n$  is even for  $X = H$  and odd for  $X = E, F$ . Similarly, for  $b_{n-1}^X$  and  $b_{n+1}^X$

$$\begin{aligned} b_{n-1}^X &= \mathcal{R}_b b_{n+1}^X \quad , \quad \mathcal{R}_b = \frac{1}{2}(u + \partial_x^{-1}u\partial_x + \frac{1}{2}\partial_x^2), \\ b_{n+1}^X &= \mathcal{R}_b^{-1} b_{n-1}^X \quad , \quad \mathcal{R}_b^{-1} = 2\partial_x^{-1} - \frac{1}{2}\mathcal{R}_a^{-1}\partial_x, \end{aligned} \quad (2.33)$$

in which  $n$  is even for  $X = H$  and odd for  $X = E, F$ . The linear differential operators  $\mathcal{R}_a, \mathcal{R}_b$  are called recursive operators [15] and they generate the integrable flows of an hierarchy (next Section). We have proven here that the proper dressing transformation (2.30),(2.31) can be thought of as generated by the inverse power of the recursive operators, *i.e.*, in a compact notation,

$$\Delta_n^X \phi' = -\partial_x \mathcal{R}_a^{\frac{-n-\nu}{2}} a_\nu^X, \quad \Delta_n^X u = -2\partial_x \mathcal{R}_b^{\frac{-n-1-\nu'}{2}} b_{\nu'}^X \quad (2.34)$$

where  $\nu(H) = 0$ ,  $\nu(E) = \nu(F) = 1$  and  $\nu'(H) = 1$ ,  $\nu'(E) = \nu'(F) = 0$ .

Now we develop a general scheme to find the algebra of the infinitesimal dressing transformations (2.24) and we will use the same procedure to find the commutation relations for the whole symmetry algebra we will discuss in the next sections. The procedure is based on three steps.

**Lemma 2.1** *The equations of motion of the resolvents (2.22) under the flows (2.24) have the form :*

$$\Delta_n^X Z^Y = [\Theta_n^X, Z^Y] - \lambda^{-n} Z^{[X,Y]}. \quad (2.35)$$

*Proof.* As first step we prove that

$$\tilde{Z} = \Delta_n^X Z^Y - [\Theta_n^X, Z^Y] \quad (2.36)$$

is a resolvent (of  $\mathcal{L}$ ):

$$\begin{aligned} 0 &= \Delta_n^X [\mathcal{L}, Z^Y] = [[\Theta_n^X, \mathcal{L}], Z^Y] + [\mathcal{L}, \Delta_n^X Z^Y] = -[[Z^Y, \Theta_n^X], \mathcal{L}] - [[\mathcal{L}, Z^Y], \Theta_n^X] + \\ &+ [\mathcal{L}, \Delta_n^X Z^Y] = [\mathcal{L}, \Delta_n^X Z^Y - [\Theta_n^X, Z^Y]]. \end{aligned} \quad (2.37)$$

From definition (2.36) it has the form

$$\tilde{Z} = \sum_{l=0}^{\infty} \lambda^l (\delta_n^X Z_l^Y - \sum_{m=0}^n [Z_m^X, Z_{l+n-m}^X]) - \sum_{l=-n}^{-1} \lambda^{-l} Z_{l+n}^{[X,Y]}. \quad (2.38)$$

Now, we must distinguish two cases. In the first case  $X \neq Y$  and the first negative powers are

$$\tilde{Z} = -\lambda^{-r} Z_0^{[X,Y]} - \dots \quad (2.39)$$

But, given the first term of a resolvent, it is completely determined by the recursive relations (2.25). In the second case  $X = Y$  and the last term in the previous equation vanishes. Therefore  $\tilde{Z}$  is expressed by the series

$$\tilde{Z} = \sum_{l=0}^{\infty} \lambda^l (\delta_n^X Z_l^Y - \sum_{m=0}^n [Z_m^X, Z_{l+n-m}^X]) \quad (2.40)$$

and the first non-zero term must be a linear combination of  $Z_0^H, Z_0^E, Z_0^F$  (see the first of equations (2.25)). It is clear that this is not possible and consequently  $\tilde{Z} = 0$ , *q.e.m.*.

**Lemma 2.2** *The equations of motion of the connections (2.23) under the flows (2.24) have the form :*

$$\Delta_n^X \Theta_s^Y - \Delta_s^Y \Theta_n^X = [\Theta_n^X, \Theta_s^Y] - \Theta_{n+s}^{[X,Y]}. \quad (2.41)$$

*Proof.* By using the definition of  $\Theta_n$  (2.23) and the previous Lemma 2.1, we obtain

$$\begin{aligned} \Delta_n^X \Theta_s^Y - \Delta_s^Y \Theta_n^X &= \\ &= (\lambda^{-s} [\Theta_n^X, Z^Y])_- - (\lambda^{-n-s} Z^{[X,Y]})_- - (\lambda^{-n} [\Theta_s^Y, Z^X])_- + (\lambda^{-n-s} Z^{[Y,X]})_- = \\ &= -([\lambda^{-s} Z^Y, (\lambda^{-n} Z^X)_-])_- - ([(\lambda^{-s} Z^Y)_-, \lambda^{-n} Z^X])_- - 2\Theta_{n+s}^{[X,Y]} = \\ &= ([\lambda^{-s} Z^Y, (\lambda^{-n} Z^X)_+ - \lambda^{-n} Z^X])_- - ([(\lambda^{-s} Z^Y)_-, \lambda^{-n} Z^X])_- - 2\Theta_{n+s}^{[X,Y]} = \\ &= ([(\lambda^{-s} Z^Y)_+ + (\lambda^{-s} Z^Y)_-, (\lambda^{-n} Z^X)_+])_- - ([(\lambda^{-s} Z^Y)_-, \lambda^{-n} Z^X])_- - \Theta_{n+s}^{[X,Y]} \\ &= ([(\lambda^{-s} Z^Y)_-, (\lambda^{-n} Z^X)_+])_- - ([(\lambda^{-s} Z^Y)_-, \lambda^{-n} Z^X])_- - \Theta_{n+s}^{[X,Y]}, \end{aligned} \quad (2.42)$$

from which the claim follows very simply, *q.e.m.*.

**Theorem 2.1** *The algebra of the vector fields (2.24) form a representation of (twisted) Borel subalgebra  $A_1 \otimes \mathbf{C}$  (of the loop algebra  $A_1^{(1)}$ ):*

$$[\Delta_n^X, \Delta_s^Y] = -\Delta_{n+s}^{[X,Y]} \quad ; \quad X, Y = H, E, F. \quad (2.43)$$

*Proof.* We have to evaluate the action of the commutator in the l.h.s. on  $\mathcal{L}$  by using the equation of motion of  $\mathcal{L}$  (2.24), the previous Lemma (2.2) and the Jacobi identity:

$$\begin{aligned} [\Delta_n^X, \Delta_s^Y] \mathcal{L} &= \Delta_n^X [\Theta_s^Y, \mathcal{L}] - \Delta_s^Y [\Theta_n^X, \mathcal{L}] = \\ &= [\Delta_n^X \Theta_s^Y - \Delta_s^Y \Theta_n^X, \mathcal{L}] + [\Theta_s^Y, [\Theta_n^X, \mathcal{L}]] - [\Theta_n^X, [\Theta_s^Y, \mathcal{L}]] = \\ &= -[\Theta_{n+s}^{[X,Y]}, \mathcal{L}], \end{aligned} \quad (2.44)$$

which is exactly the claim, *q.e.m.*.

To get from the previous Theorem 2.1 the usual form of the algebra, it is enough to undertake the replacement  $\Delta \rightarrow -\Delta$  and untwist. This kind of transformations are historically called dressing transformations [4]. In consideration of the fact that all our symmetries will be obtained by dressing, we will call them *proper* dressing transformations (or flows). In the case of mKdV, these flows are non local except the first ones which have the form of a Liouville model equation of motion:

$$\Delta_1^E \phi'(x) = e^{2\phi(x)} \quad , \quad \Delta_1^F \phi'(x) = -e^{-2\phi(x)}. \quad (2.45)$$

In particular, the Theorem 2.1 means that these infinitesimal variations (2.45) generate by successive commutations all the proper dressing flows. In addition, from them it is simple to get the Sine-Gordon equation in light-cone coordinates  $x_{\pm}$  for the boson

$$\phi \rightarrow \frac{i}{2} \phi, \quad (2.46)$$

if we define

$$x_- = x \quad , \quad \frac{\partial}{\partial x_+} = \frac{1}{2i}(\Delta_1^E + \Delta_1^F). \quad (2.47)$$

Indeed, it comes from (2.45) that

$$\partial_+ \partial_- \phi = \sin \phi \quad (2.48)$$

where it has been defined  $\partial_{\pm} = \frac{\partial}{\partial x_{\pm}}$ .

The currents, originating from this symmetry algebra, can easily be found by applying the transformations (2.30) to both members of the continuity equation (2.1)

$$J_{t,n}^X = \partial_x a_n^X(x) \quad , \quad J_{x,n}^X = \Delta_n^X \left( -\frac{1}{2}v^3 - \frac{1}{4}v'' \right). \quad (2.49)$$

To the  $J_{t,n}^X$  correspond the non-local charges

$$Q_n^X = \int_0^L J_{t,n}^X = a_n^X(L), \quad (2.50)$$

which are not necessarily conserved (depending on the boundary conditions), due to non-locality.

It is possible to verify by explicit calculations or from the Poisson brackets (2.14) that the charges themselves close a (twisted) Borel subalgebra  $A_1 \otimes \mathbf{C}$  (of the loop algebra  $A_1^{(1)}$ ).

It is interesting to note that the action by which these charges generate the transformations (2.30) is not always symplectic, but only in the case of the variations  $\Delta_1^E, \Delta_1^F$ . For instance the following Poisson brackets

$$\begin{aligned} \Delta_1^E v &= \{Q_1^E, v\} \quad , \quad \Delta_1^F v = \{Q_1^F, v\} \\ \Delta_2^H v &= \{Q_2^H, v\} + Q_1^E \{Q_1^F, v\} - Q_1^E \{Q_1^F, v\} \end{aligned} \quad (2.51)$$

denote how in the first case the action is symplectic, while in the second it is of Poisson-Lie type [4].

As a matter of fact, we will compute in the next Section how the transformations (2.24) act on  $\partial_t - A_t$  (and the other higher time Lax operators of the hierarchy), finding that they do as a gauge transformations.

## 2.3 The Integrable Hierarchy and the Asymptotic Dressing.

It is however well-known [13, 5] that besides the regular expansion of the transfer matrix an asymptotic expansion exists for the latter. Since this will play an essential role in our construction, we shall review a few important points in the procedure to obtain the asymptotic expansion. The main idea is to apply a gauge transformation  $S(x)$  on the Lax operator  $\mathcal{L}$  in such a way that its new connection  $D(x; \lambda)$  will be diagonal :

$$(\partial_x - A_x(x))S(x) = S(x)(\partial_x + D(x)). \quad (2.52)$$

Because of the previous equation  $T(x; \lambda)$  takes the form

$$T(x; \lambda) = KG(x; \lambda)e^{-\int_0^x dy D(y)} \quad (2.53)$$

where we put  $S = KG$  with

$$K = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (2.54)$$

while  $G$  verifies the following equation

$$\partial_x G + \tilde{A}_x G = GD \quad , \quad \tilde{A}_x = K^{-1} A_x K. \quad (2.55)$$

It is clear now that the previous equation can be solved by finding the asymptotic expansion for  $D(x; \lambda)$

$$D(x; \lambda) = \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix} = \sum_{i=-1}^{\infty} \lambda^{-i} d_i(x) H^i, \quad (2.56)$$

and expressing the asymptotic expansion of  $G(x; \lambda)$  in terms of off-diagonal matrices

$$G(x; \lambda) = \begin{pmatrix} 1 & g_+ \\ g_- & -1 \end{pmatrix} = H + \sum_{j=1}^{\infty} \lambda^{-j} G_j(x), \quad (2.57)$$

where the matrices  $G_j(x)$  are off-diagonal with entries

$$(G_j(x))_{12} = g_j(x) \quad , \quad (G_j(x))_{21} = (-1)^{j+1} g_j(x). \quad (2.58)$$

In addition, the off-diagonal part,  $g_j(x)$ , can be separated obeying an equation of Riccati type. The latter is solved by a recurrence formula for the  $g_j(x)$

$$g_1 = -\frac{v}{2} \quad , \quad g_{j+1} = \frac{1}{2} (g_j' + v \sum_{k=1}^{j-1} g_{i-j} g_j). \quad (2.59)$$

In addition, it is simple to see that the diagonal part  $d_j(x)$ ,  $j > 0$ , is related to  $g_j(x)$  by

$$d_j = (-1)^{j+1} v g_j, \quad (2.60)$$

and is given by  $d_{-1} = -1$ ,  $d_0 = 0$ . Note that the  $d_{2n}(x)$  are exactly the charge densities (of the mKdV equation) resulting from the asymptotic expansion of

$$\tau(\lambda) = \text{tr} M(\lambda), \quad M(\lambda) = T(L, \lambda) G^{-1}(0, \lambda) K^{-1}, \quad (2.61)$$

if we impose quasi-periodic boundary conditions on  $\phi$ .

On the other hand, it is likewise known [5] that the construction of the mKdV flows goes through the definition of the asymptotic expansion of a resolvent

$$Z^H(x, \lambda) = \sum_{k=0}^{\infty} \lambda^{-k} Z_k^H, \quad Z_0^H = H. \quad (2.62)$$

defined through the following property

$$[\mathcal{L}, Z^H(x; \lambda)] = 0. \quad (2.63)$$

The previous equation may be translated into a recursive system of differential equations for the entries of  $Z^H(x; \lambda)$  and the solution turns out to have the form

$$Z_{2k}^H(x) = b_{2k}(x)E + c_{2k}(x)F \quad , \quad Z_{2k+1}^H(x) = a_{2k+1}(x)H, \quad (2.64)$$

where

$$\begin{aligned} a_{2k+1} &= \phi' b_{2k} - \frac{1}{2} b_{2k}' \\ a_{2k+1} &= \phi' c_{2k} + \frac{1}{2} c_{2k}' \\ a_{2k-1}' &= c_{2k} - b_{2k}. \end{aligned} \quad (2.65)$$

In a way similar to what has been done for proper dressing transformation, we build through  $Z^H$  the hierarchy of commuting mKdV flows defining the gauge connections

$$\theta_{2k+1}^H(x; \lambda) = (\lambda^{2k+1} Z^H(x; \lambda))_+ = \sum_{j=0}^{2k+1} \lambda^{2k+1-j} Z_j^H(x), \quad k \in \mathbb{N} \quad (2.66)$$

and their induced transformation

$$\delta_{2k+1}^H A_x = -[\theta_{2k+1}^H(x; \lambda), \mathcal{L}]. \quad (2.67)$$

The form of  $Z_{2k+1}^H$  given by equation (2.64) imposes the self-consistency requirement  $[\theta_n^H(x; \lambda), \mathcal{L}] \propto H$  satisfied only for odd subscript  $n = 2k + 1$ .

The action (2.67) of the mKdV-flows on the bosonic field can be re-cast in terms of  $a_{2k+1}$  by using the recursive system (2.65)

$$\delta_{2k+1} \phi' = \partial_x a_{2k+1}. \quad (2.68)$$

Instead, using the previous equations of motion and the relations (2.65) it is simple to show that the action on the *classical stress-energy tensor*  $u(x)$  (2.10) is given in terms of  $b_{2k}(x)$

$$\delta_{2k+1} u = 2 \partial_x b_{2k+2}. \quad (2.69)$$

These relations have a form similar to that of proper dressing symmetries and consequently we are again interested in separating the recursive system (2.65) into one single recursion relation for  $a_{2k+1}$ . It is simple to show that the desired equation involves exactly the same recursion operator  $\mathcal{R}_a$  of equation (2.32):

$$a_{2k+1}(x) = \mathcal{R}_a a_{2k-1}(x) \quad , \quad \mathcal{R}_a = -v \partial_x^{-1} v \partial_x + \frac{1}{4} \partial_x^2. \quad (2.70)$$

This equation determines uniquely  $a_{2k+1}(x)$ , once the initial value of  $a_1(x)$  has been given. For a similar reason we obtain a recursive differential equation for  $b_{2k+2}(x)$

$$b'_{2k+2}(x) = \frac{1}{2}u'b_{2k} + ub'_{2k} + \frac{1}{4}b'''_{2k}. \quad (2.71)$$

This equation determines uniquely  $b_{2k+2}(x)$ , once the initial value of  $b_0$  has been given. Indeed it implies

$$b_{2k+2} = \mathcal{R}_b b_{2k}, \quad (2.72)$$

where  $\mathcal{R}_b$  is the same as in equation (2.33). The arbitrariness in the initial condition for  $a_{2k+1}$  and  $b_{2k}$  will be fixed in the following using the geometrical interpretation of the resolvent (equation (2.74)). The recursive operators  $\mathcal{R}_a, \mathcal{R}_b$  [15] generate the integrable flows of the hierarchy as implied by (2.67,2.68):

$$\delta_{2k+1}\phi' = \partial_x \mathcal{R}_a^k \phi' \quad , \quad \delta_{2k+1}u = 2\partial_x \mathcal{R}_b^k \cdot 1. \quad (2.73)$$

Now, like in the previous Section, it is interesting to interpret this solution  $Z^H$  to equation (2.63) as generated by dressing through the asymptotic expansion of  $T(x, \lambda)$  (2.53),(2.56),(2.60)

$$Z^H(x, \lambda) = (THT^{-1})(x, \lambda). \quad (2.74)$$

The previous similarity transformation fixes the initial conditions

$$a_1(x) = \phi' \quad , \quad b_0 = 1 \quad (2.75)$$

throughout which all the other  $a_{2k+1}$  and  $b_{2k}$  can be determined via (2.72) and (2.70). As a consequence of this fact the  $b_{2k}$  are the densities of the LIMI. Indeed, the differential relation (2.71) (or equivalently (2.72)) coincides (up to a normalization factor of  $u$ ) with that satisfied by the expansion modes of the diagonal of the resolvent of the Sturm-Liouville operator  $\partial_x - u$  [16]. The initial condition  $b_0 = 1$  makes the  $b_{2k}$  proportional to the aforementioned modes [16].

The observation (2.74) makes evident the same geometrical origin of integrable hierarchies and of their proper dressing symmetries. In addition it will allow us in the sequel to build a more general kind of symmetries and find out their algebra. Indeed, the first generalization of (2.74) consists in the construction of the flows deriving from the resolvents

$$Z^E(x, \lambda) = (TET^{-1})(x, \lambda) \quad , \quad Z^F(x, \lambda) = (FTF^{-1})(x, \lambda). \quad (2.76)$$

Unlike the previous case, these resolvents possess an expansion in all the powers of  $\lambda$

$$Z^E(x, \lambda) = \sum_{i=-\infty}^{+\infty} \lambda^{-i} Z_i^E \quad , \quad Z^F(x, \lambda) = \sum_{j=-\infty}^{+\infty} \lambda^{-j} Z_j^F. \quad (2.77)$$

In terms of the data (2.56),(2.57) of the asymptotic transfer matrix, they take the form

$$Z^E(x; \lambda) = \frac{1}{2(1 + g_+ g_-)} e^{-2I(x)} \begin{pmatrix} g_-^2 - 1 & (g_- + 1)^2 \\ -(g_- - 1)^2 & 1 - g_-^2 \end{pmatrix} \quad (2.78)$$

and

$$Z^F(x; \lambda) = \frac{1}{2(1 + g_+ g_-)} e^{2I(x)} \begin{pmatrix} g_+^2 - 1 & -(g_+ - 1)^2 \\ (g_+ + 1)^2 & 1 - g_+^2 \end{pmatrix}, \quad (2.79)$$

after defining the function

$$I(x) = -\frac{1}{2} \int_0^x (d_-(y) - d_+(y)) dy = \sum_{k=-1}^{\infty} \lambda^{-2k-1} \int_0^x d_{2k+1}(y) dy, \quad (2.80)$$

which generates the LIMT once calculated in  $x = L$ . Now, it is easy one to convince himself that the entries of the resolvents (2.78),(2.79) admit an expansion in *all* the (positive and negative) powers of  $\lambda$ , *i.e.* that the modes  $Z_i^E$  and  $Z_j^F$  of the series (2.77) are made up of linear combinations of *all* the three Lie algebra generators  $E, F, H$ . This implies the impossibility to satisfy the self-consistency condition  $\delta \mathcal{L} \propto H$ . Nevertheless, we can go over this difficulty by defining two other resolvents, combinations of the previous ones

$$Z^+(x, \lambda) = (T(E + F)T^{-1})(x, \lambda) \quad , \quad Z^-(x, \lambda) = (T(E - F)T^{-1})(x, \lambda). \quad (2.81)$$

By using the expressions (2.78),(2.79), we obtain the following formulæ for the entries of  $Z^\pm$

$$\begin{aligned} (Z^+)_{11} &= \frac{1}{2(1 + g_+ g_-)} [(g_-^2 - g_+^2 - 2) \cosh 2I + (g_+^2 - g_-^2) \sinh 2I], \\ (Z^+)_{12} &= \frac{1}{2(1 + g_+ g_-)} [(g_-^2 - g_+^2 + 2g_- + 2g_+) \cosh 2I - \\ &\quad - (g_-^2 + g_+^2 - 2g_- - 2g_+ + 2) \sinh 2I], \\ (Z^+)_{21} &= \frac{1}{2(1 + g_+ g_-)} [(g_+^2 - g_-^2 + 2g_- + 2g_+) \cosh 2I + \\ &\quad + (g_-^2 + g_+^2 - 2g_- + 2g_+ + 2) \sinh 2I], \\ (Z^+)_{22} &= -(Z^+)_{11}; \end{aligned} \quad (2.82)$$

and

$$\begin{aligned} (Z^-)_{11} &= \frac{1}{2(1 + g_+ g_-)} [(g_-^2 - g_+^2) \cosh 2I - (g_+^2 + g_-^2 - 2) \sinh 2I], \\ (Z^-)_{12} &= \frac{1}{2(1 + g_+ g_-)} [(g_-^2 + g_+^2 + 2g_- - 2g_+ + 2) \cosh 2I - \\ &\quad - (g_+^2 - g_-^2 - 2g_- - 2g_+) \sinh 2I], \\ (Z^-)_{21} &= \frac{1}{2(1 + g_+ g_-)} [-(g_+^2 + g_-^2 + 2g_+ - 2g_- + 2) \cosh 2I + \\ &\quad + (g_-^2 - g_+^2 - 2g_- - 2g_+) \sinh 2I], \\ (Z^-)_{22} &= -(Z^-)_{11}. \end{aligned} \quad (2.83)$$

If we assume to denote by  $(e)$  and  $(o)$  series with only even and odd powers of  $\lambda$  respectively, we have

$$\begin{aligned} g_+ g_- &= (e) & , & & g_+^2 + g_-^2 &= (e) & , & & g_+^2 - g_-^2 &= (o), \\ g_+ + g_- &= (o) & , & & g_+ - g_- &= (e). \end{aligned} \quad (2.84)$$

Consequently, the parity of the entries of  $Z^\pm$  is given by

$$Z^+ = \begin{pmatrix} (e) & (o) \\ (o) & (e) \end{pmatrix} \quad , \quad Z^- = \begin{pmatrix} (o) & (e) \\ (e) & (o) \end{pmatrix}, \quad (2.85)$$

or equivalently by

$$\begin{aligned} Z^+ &= \sum_{i=-\infty}^{+\infty} \lambda^{-2i-1} (b_{2i+1}^+ E + c_{2i+1}^+ F) + \sum_{i=-\infty}^{+\infty} \lambda^{-2i} a_{2i}^+ H \\ Z^- &= \sum_{j=-\infty}^{+\infty} \lambda^{-2j} (b_{2i}^- E + c_{2i}^- F) + \sum_{j=-\infty}^{+\infty} \lambda^{-2j-1} a_{2i+1}^- H. \end{aligned} \quad (2.86)$$

It follows that self-consistency requirement may now be satisfied and we are allowed to define two new series of dressing transformations through the connections

$$\begin{aligned} \theta_{2i}^+(x; \lambda) &= (\lambda^{2i} Z^+(x; \lambda))_+ = \sum_{l=-\infty}^{2i} \lambda^{2i-l} Z_l^+(x), \quad i \in \mathbb{Z} \\ \theta_{2j+1}^-(x; \lambda) &= (\lambda^{2j+1} Z^-(x; \lambda))_+ = \sum_{l=-\infty}^{2j+1} \lambda^{2j+1-l} Z_l^-(x), \quad j \in \mathbb{Z} \end{aligned} \quad (2.87)$$

which are no more finite sums. Finally, the following gauge transformations of the Lax operator  $\mathcal{L}$

$$\delta_{2i}^+ A_x = -[\theta_{2i}^+(x; \lambda), \mathcal{L}] \quad , \quad \delta_{2j+1}^- A_x = -[\theta_{2j+1}^-(x; \lambda), \mathcal{L}] \quad (2.88)$$

yield this compact form for the additional mKdV flows

$$\delta_{2i} \phi' = \partial_x a_{2i}^+ \quad , \quad \delta_{2j+1} \phi' = \partial_x a_{2j+1}^- . \quad (2.89)$$

These flows are complicated series in  $x$  with quasi-local coefficients, so that it would be very difficult to find their commutation relations by direct computation.

Therefore, to find the algebra of these additional dressing transformations (2.88), we use the previous procedure based on three steps.

**Lemma 2.3** *The equations of motion of the resolvents (2.81), (2.74) under the flows (2.88) have the form :*

$$\delta_n^X Z^Y = [\theta_n^X(x; \lambda), Z^Y] - \lambda^n Z^{[X,Y]}, \quad (2.90)$$

where now  $X, Y = H, E + F, E - F$ .

*Proof.* We omit the specific proof because it can be carried out along the lines of the analogous Lemma (2.1).

**Lemma 2.4** *The equations of motion of the connections (2.66),(2.87) under the flows (2.88) have the form :*

$$\delta_n^X \theta_s - \delta_s^Y \theta_n = [\theta_n^X, \theta_s^Y] - \theta_{n+s}^{[X,Y]}. \quad (2.91)$$

*Proof.* The proof is analogous to that of Lemma (2.2).

**Theorem 2.2** *The algebra of the asymptotic dressing vector fields (2.67),(2.88) is:*

$$\begin{aligned} [\delta_{2k+1}^H, \delta_{2i}^+] &= -2\delta_{2k+2i+1}^-, & k \in \mathbb{N}, & i \in \mathbb{Z}, \\ [\delta_{2k+1}^H, \delta_{2j+1}^-] &= -2\delta_{2k+2j+2}^+, & j \in \mathbb{Z}, \\ [\delta_{2k+1}^H, \delta_{2l+1}^H] &= 0, & l \in \mathbb{N}, \\ [\delta_{2i}^+, \delta_{2j+1}^-] &= 2\delta_{2i+2j+1}^H, \end{aligned} \quad (2.92)$$

where in the last relation we have defined  $\delta_{2k+1}^H = 0$  if  $k < 0$ .

*Proof.* As in the proof of Theorem (2.1), the action on  $\mathcal{L}$  of the commutators in the l.h.s. can be calculated by using the equation of motion of  $\mathcal{L}$  (2.88),(2.67), the previous Lemma (2.4) and the Jacobi identity, *q.e.m.*.

The previous Theorem 2.2 proves that the KdV flows form an hierarchy ( they commute with each other). Besides, they are local and the Lemma 2.4 ensures that each Lax connection transforms in a gauge way under a generic flow. This is why we may attach a *time*  $t_k$  to each flow  $\delta_{2k+1}^H$  and think to each flow as a true symmetry of all the others. Instead, the additional asymptotic flows  $\delta_{2i}^+$  and  $\delta_{2j+1}^-$  do not commute with the hierarchy flows, but close an algebra in which they are (in some sense) *spectrum generating symmetries*.

It is easy to prove that the proper dressing transformation are true symmetries of the hierarchy as well.

**Lemma 2.5** *The transformation of the resolvent (2.74) under the regular flows (2.24) and the evolution with the times  $t_k$  of the regular resolvents (2.22) have the same form of the hierarchy flow of  $\mathcal{L}$ :*

$$\delta_n^X Z^H = [\Theta_n^X, Z^H] \quad , \quad \delta_{2k+1}^H Z^X = [\theta_{2k+1}^H, Z^X], \quad (2.93)$$

where for the regular resolvents we have  $X = H, F, F$ .

**Lemma 2.6** *The mKdV flows (2.67) act as gauge transformations on the connections (2.23) of the proper dressing flows:*

$$\delta_{2k+1}^H \Theta_n - \Delta_n^X \theta_{2k+1} = [\theta_{2k+1}^H, \Theta_n^X]. \quad (2.94)$$

**Theorem 2.3** *The proper dressing vector fields (2.24) commute with the mKdV flows (2.67):*

$$[\delta_{2k+1}^H, \Delta_n^X] = 0. \quad (2.95)$$

In particular, the previous Theorem 2.3 implies that the light cone evolution  $\partial_+$  commutes with all the KdV flows, *i.e.* a different way to say that the KdV hierarchy is a symmetry of the light-cone SG. In particular, the symmetry generator  $\delta_{2k+1}^H$  maps, at infinitesimal level, solution of SG into solution.

In consideration of the fact that these theories are classical limits of CFT's and PCFT's, let us concentrate our attention on the phase spaces of mKdV and KdV systems, *i.e.* those objects which at the quantum level constitute the *spectrum of fields*.

### 3 The spectrum of fields in the $A_1^{(1)}$ framework.

In the mKdV theory a local field is a polynomial in  $v = -\phi'$  and its derivatives and the space spanned by these polynomials is the space (Verma module) of the descendant of the identity. Instead, the action (simple product in the classical theory) of these polynomials on a primary field  $e^{m\phi}$  generates the space (Verma module) of the descendants of this primary field. In our approach to the spectrum of this classical limit of CFT's, we propose here to treat the gauge fields, *i.e.* the entries of  $Z^H$  as *fundamental* fields. Let us start by considering the composite fields  $a_{2n+1}$ ,  $b_{2n}$ , and  $c_{2n}$  of (2.64). In this Section we will suppress the index H. The differential equations (2.65) tell us immediately that not all of them are independent, we may for example express the  $c_{2n}$  in terms of the *basic* fields  $b_{2n}$  and  $a_{2n+1}$ . We use now Lemma 2.3

$$\delta_{2k+1}Z = [\theta_{2k+1}, Z] \quad (3.1)$$

which allows us to establish the action of each  $\delta_{2k+1}$  on these fields

$$\begin{aligned} \delta_{2k+1}a_{2n+1} &= \sum_{i=0}^n (a'_{2n+2k-2i+1}b_{2i} - a'_{2i-1}b_{2n+2k-2i+2}), \\ \delta_{2k+1}b_{2n} &= 2 \sum_{i=0}^{n-1} (a_{2n+2k+1}b_{2n+2k-2i} - a_{2n+2k-2i+1}b_{2i}) \end{aligned} \quad (3.2)$$

Therefore, according to our conjecture the linear generators of the mKdV identity Verma module  $\mathcal{V}_1^{mKdV}$  are made up of the repeated actions of the  $\delta_{2k+1}$  on the polynomials  $\mathcal{P}(b_2, b_4, \dots, b_{2N}, a_1, a_3, \dots, a_{2P+1})$  in the  $b_{2n}$  and the  $a_{2k+1}$

$$\mathcal{V}_1^{mKdV} = \{\text{linear combinations of } \delta_{2k_1+1}\delta_{2k_2+1}\dots\delta_{2k_M+1}\mathcal{P}\}, \quad (3.3)$$

with a natural gradation provided by the subscripts. Actually, the Verma module  $\mathcal{V}_1^{mKdV}$  exhibits several null vectors, i.e. polynomials in the  $b_{2n}$  and the  $a_{2k+1}$  which are zero. This is due to the very simple constraint on  $Z^H$

$$(Z^H)^2 = \mathbf{1} \quad (3.4)$$

originating from the dressing relation with the transfer matrix  $T$  (2.74). The constraints (3.4) may be rewritten through the modes of  $a_{2k+1}$  and  $b_{2k}$

$$\mathcal{C}_{2n} = \sum_{i=0}^n b_{2n-2i}(b_{2i} + a'_{2i-1}) + \sum_{i=0}^{n-1} a_{2n-2i-1}a_{2i+1} = 0, \quad (3.5)$$

and produce null-vectors under the application of mKdV flows  $\delta_{2k+1}$ . These latter generate linearly the graded vector space (Verma module) of all null vectors

$$\mathcal{N} = \{\text{linear combinations of } \delta_{2k_1+1}\delta_{2k_2+1}\delta_{2k_3+1}\dots\delta_{2k_Q+1}\mathcal{C}_{2n}\}. \quad (3.6)$$

In conclusion our conjecture is that the (*conformal*) family of the identity  $[\mathbf{1}]^{mKdV}$  of the mKdV hierarchy is obtained as a factor space:

$$[\mathbf{1}]^{mKdV} = \mathcal{V}_1^{mKdV} / \mathcal{N}. \quad (3.7)$$

On the other hand, in order to deduce the form of the Verma module  $\mathcal{V}_1^{KdV}$  of the identity for the KdV hierarchy we have to make three observations:

1. the recursive formula (2.72) proves that  $b_{2n}$  are polynomials of the KdV field  $u(x)$  and its derivatives, whereas the  $a_{2k+1}$  do not enjoy this property;
2. the variation of  $b_{2n}$  in (3.2) can be written *accidentally* in terms of the  $b_{2k}$  alone, using the relationships (2.65) between  $a_{2k+1}$  and  $b_{2k}$

$$\delta_{2k+1}^H b_{2n} = \sum_{i=0}^{n-1} (b'_{2n+2k-2i} b_{2i} - b'_{2i} b_{2n+2k-2i}); \quad (3.8)$$

3. also the null vector space  $\mathcal{N}$  can be spanned by the  $b_{2k}$  alone

$$\mathcal{C}_{2n} = b_{2n} + \sum_{i=1}^n [b_{2n-2i} b_{2i} - 2b_2 b_{2n-2i} b_{2i-2} - \frac{1}{2} b_{2n-2i} b'_{2j-2} + \frac{1}{4} b'_{2n-2j} b'_{2j-2}] = 0, \quad (3.9)$$

using the relationships (2.65) between  $a_{2k+1}$  and  $b_{2k}$ .

Therefore, we conjecture that the Verma module  $\mathcal{V}_1^{KdV}$  shall be linearly generated by elements given by repeated actions of the  $\delta_{2k+1}$  on the polynomials  $\mathcal{P}(b_2 b_4 \dots b_{2N})$  in the  $b_{2n}$

$$\mathcal{V}_1^{KdV} = \{\text{linear combinations of } \delta_{2k_1+1}\delta_{2k_2+1}\dots\delta_{2k_M+1}\mathcal{P}\}. \quad (3.10)$$

It turns out to be a sort of *reduction* of the Verma module  $\mathcal{V}_{\mathbf{1}}^{mKdV}$  (3.3) of the mKdV hierarchy. As for the mKdV case the (*conformal*) family of the identity  $[\mathbf{1}]^{KdV}$  of the KdV hierarchy is obtained as a factor space of  $\mathcal{V}_{\mathbf{1}}^{KdV}$  over  $\mathcal{N}$  given by (3.6) and (3.9):

$$[\mathbf{1}]^{KdV} = \mathcal{V}_{\mathbf{1}}^{KdV} / \mathcal{N}. \quad (3.11)$$

Therefore we are led to the same scenario that arises also in the classical limit of the construction [3]. Nevertheless, in our approach the generation of null-vectors is automatic and geometrical (see equations (3.8) and (3.9)). In addition, our approach is applicable to any other integrable system, based on a Lax pair formulation. We will illustrate this fact below by using the example of the  $A_2^{(2)}$ -mKdV system. Other local fields of the mKdV system are the primary fields, i.e. the exponential  $e^{m\phi}$ ,  $m = 0, 1, 2, 3, \dots$  of the bosonic field. Indeed, for  $m = 0$  we obtain just the identity  $\mathbf{1}$ , the *fundamental primary field*  $e^\phi$  ( $m=1$ ) appears in the regular expansion (2.19) of the transfer matrix  $T(x; \lambda)$  and the other primary fields  $e^{m\phi}$ ,  $m > 1$  are the ingredients of the regular expansion of the power  $T^m(x; \lambda)$ . The previous construction of the identity operator family suggests the following form for the Verma module  $\mathcal{V}_m^{mKdV}$  of the primary  $e^{m\phi}$ ,  $m = 0, 1, 2, 3, \dots$ :

$$\mathcal{V}_m^{mKdV} = \{\text{linear combinations of } \delta_{2k_1+1} \delta_{2k_2+1} \dots \delta_{2k_M+1} [\mathcal{P} e^{m\phi}]\}, \quad (3.12)$$

where  $\mathcal{P}(b_2, b_4, \dots, b_{2N}, a_1, a_3, \dots, a_{2P+1})$  are polynomials in the  $b_{2n}$  and the  $a_{2k+1}$ . As for the identity family, we have to subtract all the null-vectors (3.2) and (3.9). Besides, in this case, we have to take into account the null-vectors coming from the equations of motion of the power  $T_{reg}^m(x; \lambda)$  of the regular expansion

$$\delta_{2k+1} T_{reg}^m = \sum_{j=1}^m T_{reg}^j \theta_{2k+1} T_{reg}^{m-j}. \quad (3.13)$$

By successive applications of (3.2), (3.9) and (3.13) we obtain the whole null-vector set  $\mathcal{N}_{\mathbf{m}}^{KdV}$ . In conclusion the spectrum is again the factor space

$$[\mathbf{m}] = \mathcal{V}_m^{mKdV} / \mathcal{N}_{\mathbf{m}}^{KdV}. \quad (3.14)$$

Similarly, the construction of the identity operator family suggests the following form for the Verma module  $\mathcal{V}_m^{KdV}$  of the primary  $e^{m\phi}$ ,  $m = 0, 1, 2, 3, \dots$ :

$$\mathcal{V}_m^{KdV} = \{\text{linear combinations of } \delta_{2k_1+1} \delta_{2k_2+1} \dots \delta_{2k_M+1} [\mathcal{P}(b_2, b_4, \dots, b_{2N}) e^{m\phi}]\}, \quad (3.15)$$

with  $\mathcal{P}(b_2, b_4, \dots, b_{2N})$  polynomials in  $b_{2n}$ . Again, by successive applications of (3.8), (3.9) and (3.13) we obtain the whole null-vector linear space  $\mathcal{N}_{\mathbf{m}}^{KdV}$ . In conclusion, the spectrum is again a factor space

$$[\mathbf{m}] = \mathcal{V}_m^{KdV} / \mathcal{N}_{\mathbf{m}}^{KdV}. \quad (3.16)$$

Of course, we have checked all our conjectures up to high gradation of the null vectors. Nevertheless, we did not manage to generate the spectrum of fields only through the asymptotic symmetry of Theorem 2.2. For dimensional arguments it is plausible to make the substitution

$$\begin{aligned} a_{2k+1} &\rightarrow \delta_{2k+1}^+, & k \in \mathbb{Z} \\ b_{2k} &\rightarrow \delta_{2k}^-, \end{aligned} \quad (3.17)$$

but now the null vector meaning and origin should be completely different.

## 4 A non local Virasoro symmetry by dressing.

At this point we are in a position to construct in a natural way more general kinds of dressing-like symmetries. It is well known that the vector fields  $l_m = \lambda^{m+1} \partial_\lambda$  on the circumference realize the centerless Virasoro algebra

$$[l_m, l_n] = (m - n) l_{m+n}. \quad (4.1)$$

A very natural dressing is represented by the resolvents

$$\begin{aligned} Z_m^V &= T_{reg} l_m T_{reg}^{-1}, & m < 0 \\ Z_m^V &= T_{asy} l_m T_{asy}^{-1}, & m \geq 0 \end{aligned} \quad (4.2)$$

where we have to use the different regular and asymptotic transfer matrices,  $T_{reg}$  and  $T_{asy}$ . Of course, they satisfy the usual definition of resolvent

$$[\mathcal{L}, Z_m^V(x; \lambda)] = 0 \quad (4.3)$$

and, as in the previous cases, they have two different kinds of expansions

$$\begin{aligned} (Z_{-1}^V)_{reg} &= T_{reg} l_{-1} T_{reg}^{-1} = \sum_{n=0}^{\infty} \lambda^n Z_{n+1}^{reg} - \partial_\lambda \\ (Z_{-1}^V)_{asy} &= T_{asy} l_{-1} T_{asy}^{-1} = \sum_{n=0}^{\infty} \lambda^{-n} Z_{n-1}^{asy} - \partial_\lambda \end{aligned} \quad (4.4)$$

and consequently the mode expansion of the more general Virasoro resolvent (4.2). In the same way, (4.3) authorizes us to define gauge connections

$$\begin{aligned} \theta_m^V &= (Z_m^V)_- = \sum_{n=0}^{-m-2} \lambda^{n+1+m} Z_{n+1}^{reg} - \lambda^{m+1} \partial_\lambda, & m < 0 \\ \theta_m^V &= (Z_m^V)_+ = \sum_{n=0}^{m+1} \lambda^{m+1-n} Z_{n-1}^{asy} - \lambda^{m+1} \partial_\lambda, & m \geq 0 \end{aligned} \quad (4.5)$$

and the relative gauge transformations

$$\delta_m^V A_x = -[\theta_m^V(x; \lambda), \mathcal{L}]. \quad (4.6)$$

Finally, we have to verify the consistency of this gauge transformation requiring  $\delta_m^V A_x = H\delta_m^V \phi'$  for positive and negative  $m$ . It is very easy to see that this requirement imposes  $m$  to be even. Indeed, from (4.3) or (4.2) it is simple to derive the form of the generic term of the expansions (4.4)

$$\begin{aligned} Z_{2n-1}^{reg} &= b_{2n-1}^V E + c_{2n-1}^V F, & Z_{2n}^{reg} &= a_{2n}^V H, & n > 0 \\ Z_{2n-3}^{asy} &= \beta_{2n-3}^V E + \gamma_{2n-3}^V F, & Z_{2n-2}^{asy} &= \alpha_{2n-2}^V H, & n > 0. \end{aligned} \quad (4.7)$$

In addition, we can easily find recursive relations for the regular coefficients

$$b_{2k+1}^V(x) = \mathcal{R}_b^{-1} b_{2k-1}^V(x), \quad a_{2n+2}^V(x) = \mathcal{R}_a^{-1} a_{2n}^V(x), \quad (4.8)$$

and for the asymptotic coefficients

$$\beta_{2n-1}^V(x) = \mathcal{R}_b \beta_{2n-3}^V(x), \quad \alpha_{2n}^V(x) = \mathcal{R}_a \alpha_{2n-2}^V(x), \quad (4.9)$$

where the recursive operators  $\mathcal{R}_a, \mathcal{R}_b$  are given in (2.32), (2.33) and (4.2) fixes the initial conditions:

$$\begin{aligned} b_{-1}^V &= e^{2\phi} \partial_x^{-1} e^{-2\phi}, & a_2 &= \int_0^x dx_2 \int_0^{x_2} dx_1 2 \cosh[\phi(x_1) - \phi(x_2)], \\ \beta_{-1}^V &= x, & \alpha_0^V &= x\phi'. \end{aligned} \quad (4.10)$$

In conclusion the Virasoro mKdV flows are given by:

$$\begin{aligned} \delta_{2m}^V \phi' &= -\partial_x a_{-2m}^V, & m < 0 \\ \delta_{2m}^V \phi' &= \partial_x \alpha_{2m}^V, & m \geq 0, \end{aligned} \quad (4.11)$$

and explicit examples of the first flows are

$$\begin{aligned} \delta_{-2}^V \phi' &= e^{2\phi(x)} \int_0^x dy e^{-2\phi(y)} - e^{-2\phi(x)} \int_0^x dy e^{2\phi(y)} = e^{2\phi(x)} B_1 - e^{-2\phi(x)} C_1 \\ \delta_{-4}^V \phi' &= e^{2\phi(x)} (3B_3(x) - A_2(x)B_1(x)) - e^{-2\phi(x)} (3C_3(x) - D_2(x)C_1(x)) \\ \delta_{-6}^V \phi' &= e^{2\phi(x)} (5B_5(x) - 3A_4(x)B_1(x) + A_2(x)B_3(x)) \\ &\quad - e^{-2\phi(x)} (5C_5(x) - 3D_4(x)C_1(x) + D_2(x)C_3(x)) \\ \delta_0^V \phi' &= \phi' + x\phi'' \\ \delta_2^V \phi' &= 2xa'_3 + 6g_3 - 2g_1^3 + 2g'_1 \int_0^x d_1, \end{aligned} \quad (4.12)$$

where we have used a convenient notation for the entries of the regular expansion

$$T_{reg}(x, \lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (4.13)$$

with  $A = e^\phi(1 + \sum_1^\infty \lambda^{2n} A_{2n})$ ,  $B = e^\phi \sum_0^\infty \lambda^{2n+1} B_{2n+1}$ ,  $C[\phi] = B[-\phi]$  and  $D[\phi] = A[-\phi]$ . We stress that negative subscript variations have a form very similar to that

of the regular dressing flows ((2.24) with  $X = H$ )  $\Delta_{2r}^H$ . Nevertheless, in spite of the commutativity  $[\Delta_{2r}^H, \Delta_{2s}^H] = 0$  we will see that they obey instead Virasoro commutation relations. From the actions (4.12) the transformations of the classical *primary fields*  $e^\phi$  follow. For example:

$$\begin{aligned}\delta_{-2}^V e^\phi &= (D_2 - A_2)e^\phi \\ \delta_{-4}^V e^\phi &= [(3D_4 - C_3B_1) - (3A_4 - B_3C_1)]e^\phi, \\ \delta_0^V e^\phi &= (x\partial_x + \Delta)e^\phi \\ \delta_2^V e^\phi &= (2xa_3 + 2g_2 + 2g_1 \int_0^x d_1) e^\phi.\end{aligned}\tag{4.14}$$

It is understood of course that these fields are primary with respect to the usual space-time Virasoro symmetry. The actions of the variations (4.11) on the generator of this symmetry can be easily calculated as usual by means of Miura transformation (2.10) and the recursive relations implied by (4.3)

$$\begin{aligned}\delta_{2m}^V u &= -2\partial_x b_{-2m-1}^V, & m < 0 \\ \delta_{2m}^V u &= 2\partial_x \beta_{2m+1}^V, & m \geq 0.\end{aligned}\tag{4.15}$$

For example

$$\delta_2^V u = x(u''' - \frac{3}{2}uu') + u'' - 2u^2 - \frac{1}{2}u' \int_0^x u.\tag{4.16}$$

We now apply our usual procedure in three parts to find transformation equations and symmetry algebra in this case. We will omit the proofs in consideration of the fact that they are very similar to the previous ones.

**Lemma 4.1** *The equations of motion of the resolvents (4.2) under the flows (4.6) have the form :*

$$\delta_{2n}^V Z_{2m}^V = [\theta_{2n}^V, Z_{2m}^V] - (2n - 2m)Z_{2n+2m}^V, \quad m, n \in \mathbb{Z}.\tag{4.17}$$

**Lemma 4.2** *The transformations of the connections (4.5) under the flows (4.6) have the form :*

$$\delta_{2n}^V \theta_{2m}^V - \delta_{2m}^V \theta_{2n}^V = [\theta_{2n}^V, \theta_{2m}^V] - (2n - 2m)\theta_{2n+2m}^V, \quad m, n \in \mathbb{Z}.\tag{4.18}$$

**Theorem 4.1** *The algebra of the vector fields on  $\mathcal{L}$  (4.6) forms a representation of the centerless Virasoro algebra:*

$$[\delta_{2m}^V, \delta_{2n}^V] = (2m - 2n)\delta_{2m+2n}^V, \quad m, n \in \mathbb{Z},\tag{4.19}$$

after the redefinition  $\delta^V \rightarrow -\delta^V$ .

The Theorem 4.1 gives us a very non-trivial information because of the different character of the asymptotic and regular Virasoro vector fields. Indeed, the asymptotic ones are quasi-local (they can be made local after differentiating a certain number of times), the regular ones instead are essentially non-local being expressed in terms of vertex operators. In addition, it is easy to compute the most simple relations  $[\delta_0, \delta_{2n}] = -2n\delta_{2n}$ ,  $n \in \mathbb{Z}$ , which means that  $\delta_0$  counts the dimension or level. We want to stress once more that this Virasoro symmetry is different from the space-time one and is essentially non-local. The additional symmetries coming from the regular dressing are very important for applications. They complete the asymptotic ones forming an entire Virasoro algebra and provide a possibility of a central extension in the (generalized) KdV hierarchy, which is the classical limit of CFT's. However, this central term may appear only in the algebra of the hamiltonians of the above transformations, as it is for the case of CFT's as well.

With the aim of understanding the classical and quantum structure of integrable systems, we present here the complete algebra of symmetries. The Virasoro flows commute neither with the mKdV hierarchy (2.67) nor with the (proper) regular dressing flows (2.24). In fact one can show, following the lines of the three steps procedure, these statements.

**Lemma 4.3** *The equations of motion of the resolvents (4.2) under the mKdV flows (2.67) and of the mKdV resolvent (2.74) under the Virasoro flows (4.6) have the same form of the variation of  $\mathcal{L}$ :*

$$\begin{aligned}\delta_{2k+1}^H Z_{2m}^V &= [\theta_{2k+1}^H, Z_{2m}^V], \quad k \in \mathbb{N}, \quad m \in \mathbb{Z} \\ \delta_{2m}^V Z_{2k+1}^H &= [\theta_{2m}^V, Z_{2k+1}^H].\end{aligned}\tag{4.20}$$

**Lemma 4.4** *The mixed transformations of the connections (4.5) under the mKdV flows (2.67) are not of gauge type:*

$$\delta_{2k+1}^H \theta_{2m}^V - \delta_{2m}^V \theta_{2k+1}^H = [\theta_{2k+1}^H, \theta_{2m}^V] - (2k+1)\theta_{2k+2m+1}^H.\tag{4.21}$$

**Theorem 4.2** *The algebra of the hierarchy flows and of the Virasoro flows is not abelian:*

$$[\delta_{2k+1}^H, \delta_{2m}^V] = (2k+1)\delta_{2m+2k+1}^H,\tag{4.22}$$

where we have put  $\delta_{2k+1}^H = 0$  if  $k < 0$  in the r.h.s..

The content of the previous theorem is that Virasoro symmetry shifts along the KdV hierarchy.

Likewise, it may be proven that

$$[\Delta_n^X, \delta_{2m}^V] = -n\Delta_{n-2m}^X. \quad (4.23)$$

after putting the proper dressing flows (2.24) with negative  $n - 2m$ :  $\Delta_{n-2m}^X = 0$  in the r.h.s.. As a very important consequence of this fact, the light-cone Sine-Gordon flow  $\partial_+$  (2.47),(2.48) commutes with all the positive Virasoro modes, *i.e.* we have obtained a half Virasoro algebra as exact symmetry of SGM. We note again that this infinitesimal transformation is quasi-local in the boson  $\phi$ .

One remark is necessary at this stage. It is quite interesting to have a Virasoro algebra not commuting (spectrum generating) with the KdV flows, but one may transform the Virasoro flows into true symmetries commuting with the mKdV hierarchy by adding a term containing all the times  $t_{2k+1}$

$$\delta_{2m}^V \rightarrow \delta_{2m}^V - \sum_{k=1}^{\infty} (2k+1)t_{2k+1}\delta_{2m+2k+1}^H, \quad (4.24)$$

From the view point of CFT it is very difficult to give a physical meaning to these times, but from the restriction of the action of the positive part of the Virasoro algebra on  $u$  (4.15) we can check that the previous formula yields the half Virasoro algebra described in [17],[18] by using the pseudodifferential operator method. Actually, it plays an important role in the study of the matrix models where it leads to the so called Virasoro constraints:  $L_m\tau = 0, m > 0$ . Here  $\tau$  is the  $\tau$ -function of the hierarchy and is connected to the partition function of the matrix model. Moreover, it seems that it should play an important role also in the context of the Matrix String Theory [19], which is now intensively studied. Note also that these Virasoro constraints are the conditions for the highest weight state and, because we also have  $L_0\tau \propto \tau$  [18], the  $\tau$ -function is a primary state for the Virasoro algebra. But, we uncovered the negative modes of the Virasoro algebra, which build the highest weight representation over the  $\tau$ -function. We are analyzing this intriguing scenario even in off-critical theories like Sine-Gordon [20].

Let us also note that actually the symmetry of mKdV is much larger. Indeed, the differential operators  $l_{2m,2n} = \lambda^{2m+1}\partial_\lambda^{2n+1}$  close a (twisted)  $w_\infty$  which is isomorphic to its dressed version

$$\delta_{2m,2n}A_x = -[\theta_{2m,2n}(x; \lambda), \mathcal{L}], \quad (4.25)$$

where we have defined the connections and the resolvents

$$\begin{aligned} \theta_{2m,2n} &= (Z_{2m,2n}^V)_-, & Z_{2m,2n} &= T_{reg}l_{2m,2n}T_{reg}^{-1}, & m < 0 \\ \theta_{2m,2n} &= (Z_{2m,2n}^V)_+, & Z_{2m,2n} &= T_{asy}l_{2m,2n}T_{asy}^{-1}, & m \geq 0. \end{aligned} \quad (4.26)$$

In particular from the commutations of the *diagonal* differential operators  $l_{2n,2n} = \lambda^{2n+1} \partial_\lambda^{2n+1}$

$$[l_{2n,2n}, l_{2m,2m}] = 0 \quad (4.27)$$

we deduce the existence of a quasi-local hierarchy of the diagonal flows

$$[\delta_{2n,2n}, \delta_{2m,2m}] = 0, \quad n, m > 0. \quad (4.28)$$

As far as we know, this observation is new and we suggest that the diagonal flows are connected to the higher Calogero-Sutherland hamiltonian flows in their collective field theory description. This could give a geometrical explicit explanation of the misterious connection between Calogero-Sutherland systems and KdV hierarchy [21].

## 5 Generalization: the $A_2^{(2)}$ -KdV.

Let us show that our approach is easily applicable to other integrable systems. Here we consider the case of the  $A_2^{(2)}$ -KdV equation. The reason is that it can be considered as a different classical limit of the **CFT's** [8, 22]. Consider the matrix representation of the  $A_2^{(2)}$ -KdV equation:

$$\partial_t \mathcal{L} = [\mathcal{L}, A_t] \quad (5.1)$$

where

$$\mathcal{L} = \partial_x - A_x, \quad A_x = \phi' h + (e_0 + e_1), \quad (5.2)$$

and

$$e_0 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (5.3)$$

are the generators of the Borel subalgebra of  $A_2^{(2)}$  and  $A_t$  is a certain connection that can be found for example in [5]. Again, the two  $A_2^{(2)}$ -KdV equations are given by Miura transformations; the one of our interest is again  $u(x) = -\phi'(x)^2 - \phi''(x)$ . As shown before, central role is played by the transfer matrix, wich is a solution of the associated linear problem  $(\partial_x - A_x(x; \lambda))T(x; \lambda) = 0$ . The formal solution is in this case

$$T_{reg}(x, \lambda) = e^{h\phi(x)} \mathcal{P} \exp \left( \int_0^x dy (e^{-2\phi(y)} e_0 + e^{\phi(y)} e_1) \right). \quad (5.4)$$

The equation (5.4) defines  $T$  as an entire function of  $\lambda$  with an essential singularity at  $\lambda = \infty$ . The corresponding proper dressing symmetries may be worked out in a way similar to the  $A_1^{(1)}$  case, but we are mainly interested in the spectrum of local fields. As for the  $A_1^{(1)}$  case, the asymptotic expansion is easily written by following the general procedure [5]. The result is:

$$T_{asy}(x; \lambda) = \begin{pmatrix} 1 & h_1^+ & h_2^+ \\ 0 & 1 + h_3^0 & h_1^0 \\ 0 & h_2^- & 1 + h_3^- \end{pmatrix} \exp \left( - \int_0^x \sum_{i=0}^{\infty} f_i \Lambda^{-i} \right). \quad (5.5)$$

where  $h_i^\pm, h_i^0$  are certain polynomials in  $\phi'$ ,  $f_{6k}, f_{6k+2}$  are the densities of the local conserved charges of the  $A_2^{(2)}$ -KdV and  $\Lambda = e_0 + e_1$ . Complying with our approach let us introduce the asymptotic resolvents

$$\begin{aligned} Z_1 &= T_{asy} \Lambda T_{asy}^{-1}, \\ Z_2 &= T_{asy} \Lambda^2 T_{asy}^{-1} \end{aligned} \quad (5.6)$$

satisfying as before the equations (2.20)  $[\partial_x - A_x, Z_i(x; \lambda)] = 0$ ,  $i = 1, 2$  with  $A_x$  now given by (5.2) and (5.3). These have the form:

$$Z_1 = \begin{pmatrix} \lambda^{-1}a_1^{(1)} + \lambda^{-4}a_4^{(1)} + \dots & \lambda^{-2}b_2^{(1)} + \lambda^{-5}b_5^{(1)} + \dots & 1 + \lambda^{-6}b_6^{(1)} + \dots \\ 1 + \lambda^{-3}c_3^{(1)} + \dots & -2\lambda^{-4}a_4^{(1)} + \dots & \lambda^{-2}b_2^{(1)} - \lambda^{-5}b_5^{(1)} + \dots \\ \lambda^{-2}c_2^{(1)} + \dots & 1 - \lambda^{-3}c_3^{(1)} + \lambda^{-6}c_6^{(1)} + \dots & -\lambda^{-1}a_1^{(1)} + \lambda^{-4}a_4^{(1)} + \dots \end{pmatrix},$$

and

$$Z_2 = \begin{pmatrix} \lambda^{-2}a_2^{(2)} + \lambda^{-5}a_5^{(2)} + \dots & 1 + \lambda^{-3}b_3^{(2)} + \lambda^{-6}b_6^{(2)} + \dots & \lambda^{-4}b_4^{(2)} + \dots \\ \lambda^{-1}c_1^{(2)} + \lambda^{-4}c_4^{(2)} + \dots & -2\lambda^{-2}a_2^{(2)} + \dots & 1 - \lambda^{-3}b_3^{(2)} - \lambda^{-6}b_6^{(2)} + \dots \\ 1 + \lambda^{-6}c_6^{(2)} + \dots & -\lambda^{-1}c_1^{(2)} + \lambda^{-4}c_4^{(2)} + \dots & \lambda^{-2}a_2^{(2)} - \lambda^{-5}a_5^{(2)} + \dots \end{pmatrix}.$$

For example, some expressions for the fields in the entries of  $Z_i, i = 1, 2$ , as a function of  $v = -\phi'$  and its derivative, are:

$$\begin{aligned} a_1^{(1)} &= -v; \\ b_2^{(1)} &= -\frac{1}{3}v^2 + \frac{1}{3}v' \quad , \quad c_2^{(1)} = -\frac{1}{3}v^2 - \frac{2}{3}v'; \\ c_3^{(1)} &= \frac{1}{3}v^3 + \frac{1}{3}vv' - \frac{1}{3}v'' \quad , \quad b_3^{(2)} = -\frac{2}{3}vv' + \frac{1}{3}v''; \\ b_4^{(2)} &= -\frac{1}{9}v^4 + \frac{1}{9}v'^2 - \frac{2}{9}vv'' + \frac{2}{9}v'v^2 - \frac{1}{9}v''', etc.. \end{aligned} \quad (5.7)$$

The equation (5.1) is invariant under a gauge transformation of the form (2.67). This latter will be a true symmetry provided the variation is proportional to  $h$ :  $\delta A_x = h\delta\phi'$ . We construct the appropriate gauge parameters by means of the resolvents (5.6) in a way similar to what we did in the  $A_1^{(1)}$ -mKdV case:

$$\theta_{6k+1}(x; \lambda) = (\lambda^{6k+1}Z_1(x; \lambda))_+ \quad , \quad \theta_{6k-1}(x; \lambda) = (\lambda^{6k-1}Z_2(x; \lambda))_+ \quad (5.8)$$

which results in the following transformations for the  $A_2^{(2)}$ -mKdV field

$$\delta_{6k+1}\phi' = \partial_x a_{6k+1}^{(1)} \quad , \quad \delta_{6k-1}\phi' = \partial_x a_{6k-1}^{(2)}. \quad (5.9)$$

One can easily recognize in (5.9) the infinite tower of the commuting  $A_2^{(2)}$ -mKdV flows.

Now, in accordance with our geometrical conjecture, we would like to treat the entries of the transfer matrix  $T$  and of the resolvents  $Z_i, i = 1, 2$  as independent fields

and to build the spectrum of the local fields of  $A_2^{(2)}$ -KdV by means of them alone. As in the  $A_1^{(1)}$  case, it turns out that not all of them are independent. If the defining relations of the resolvents are used, it is easy to see, that the entries of the lower triangle of both  $Z_i$  can be expressed in terms of the rest. Therefore, taking also into account the gauge symmetry of the system, one is led to the following proposal about the construction of the Verma module of the identity:

$$\mathcal{V}_0^{mKdV} = \{l.c.o. \quad \delta_{6k_1+1} \dots \delta_{6k_M+1} \delta_{6l_1-1} \dots \delta_{6l_N-1} \mathcal{P}(b_i^{(1)}, b_j^{(2)}, a_k^{(1)}, a_l^{(2)})\}, \quad (5.10)$$

where *l.c.o.* means *linear combinations of*. Again, null-vectors appear in the r.h.s. of the (5.10) due to the constraints:

$$Z_1^2 = Z_2 \quad , \quad Z_1 Z_2 = \mathbf{1} \quad (5.11)$$

and the equations of motion

$$\delta_{6k\pm 1} Z_i = [\theta_{6k\pm 1}, Z_i] \quad , \quad i = 1, 2. \quad (5.12)$$

One can further realize that just as in the  $A_1^{(1)}$  case, there is a subalgebra consisting of the upper triangular entries of  $Z_i, i = 1, 2$ , closed under the action of the gauge transformations  $\delta_{6k\pm 1}$ . The constraints (5.11) and (5.12) are consistent with such reduction giving a closed subalgebra of null vectors. The first non-trivial examples are :

$$\begin{aligned} \text{level3} & : \quad b_3^{(2)} - \partial_x b_2^{(1)} = 0; \\ \text{level4} & : \quad b_4^{(2)} - (b_2^{(1)})^2 + \frac{2}{3} \partial_x b_3^{(2)} = 0; \\ \text{level6} & : \quad b_6^{(1)} - 2b_6^{(2)} + 2b_2^{(1)} b_4^{(2)} + (b_3^{(2)})^2 = 0; \\ & \quad 2 \quad b_6^{(1)} - b_6^{(2)} + 2\partial_x b_5^{(1)} + \frac{1}{2} \partial_x^2 b_4^{(2)} + b_2^{(1)} b_4^{(2)} - (b_2^{(1)})^3 + (b_3^{(2)})^2 = 0. \end{aligned} \quad (5.13)$$

Therefore, in order to obtain the true spectrum of the family of the identity (i.e. the  $A_2^{(2)}$ -KdV spectrum), one has to factor out, from the linearly generated Verma module

$$\mathcal{V}_0^{KdV} = \{l.c.o. \quad \delta_{6k_1+1} \dots \delta_{6k_M+1} \delta_{6l_1-1} \dots \delta_{6l_N-1} \mathcal{P}(b_i^{(1)}, b_j^{(2)})\}, \quad (5.14)$$

the Verma module of null-vectors  $\mathcal{N}_0^{KdV}$ , i.e.

$$[\mathbf{0}] = \mathcal{V}_0^{KdV} / \mathcal{N}_0^{KdV}. \quad (5.15)$$

Let us turn to the classical limit of the primary fields  $e^{m\phi}, m = 0, 1, 2, 3, \dots$ . By virtue of the above reasoning we conjecture for their Verma modules the expression

$$\mathcal{V}_0^{mKdV} = \{l.c.o. \quad \delta_{6k_1+1} \dots \delta_{6k_M+1} \delta_{6l_1-1} \dots \delta_{6l_N-1} [\mathcal{P}(b_i^{(1)}, b_j^{(2)}) e^{m\phi}]\}. \quad (5.16)$$

Again, we have to add to the null-vectors coming from (5.11) and (5.12) the new ones coming from (repeated) application of the equations of motion of the power  $T^m(x; \lambda)$

$$\delta_{6k\pm 1} T^m = \sum_{j=1}^m T^j \theta_{6k\pm 1} T^{m-j} \quad (5.17)$$

obtaining the whole set of null-vectors  $\mathcal{N}_{\mathbf{m}}^{KdV}$ . The first non-trivial examples of these additional null-vectors are given by:

$$\begin{aligned} \text{level } 2 & : (\partial_x^2 + 3b_2^{(1)})e^\phi = 0 \\ \text{level } 3 & : (\partial_x^3 - 6b_3^{(2)} + 12\partial_x b_2^{(1)})e^{2\phi} = 0 \\ \text{level } 4 & : (\partial_x^4 + \frac{135}{2}(b_2^{(1)})^2 + 30\partial_x^2 b_2^{(1)} - \frac{27}{2}b_4^{(2)} - 30\partial_x b_3^{(2)})e^{3\phi} = 0 \end{aligned} \quad (5.18)$$

where the operator  $\partial_x$  acts on all the fields to its right. As a result, the (*conformal*) family of the primary field  $e^{m\phi}$ ,  $m = 0, 1, 2, 3, \dots$  is conjectured to be in this case

$$[\mathbf{m}] = \mathcal{V}_m^{KdV} / \mathcal{N}_{\mathbf{m}}^{KdV}. \quad (5.19)$$

## 6 Conclusions and perspectives: off-critical theories and quantisation.

We have derived different types of symmetries of classical integrable systems within a general framework. The unifying geometrical idea bases on dressing simple objects by means of the transfer matrix  $T$  with the aim to get the resolvent  $Z$ . In particular, the regular transfer matrix gives rise to a Poisson–Lie symmetry of non-commuting flows. The construction of the spectrum employing this symmetry is an interesting open problem, clearly connected with the spinon basis defined in [2], since the current density  $e^\phi$  is exactly the spinon of [2] after quantisation. Moreover, it is of great interest the new way to look at the Sine-Gordon light-cone evolution as generated by the sum of the first two regular vector fields. It is also important to note that a similar half Virasoro symmetry can be obtained just underchanging the rôle of  $x_-$  and  $x_+$ . We leave for a work in progress the very important question on the whole algebra obtained from the union of both half Virasoro algebras [20].

On the contrary, the asymptotic formula for the transfer matrix provides the integrable hierarchy of (generalized) KdV flows and, in the case of dressing of non-cartan generators, two new infinite series of flows closing an algebraic structure with the integrable ones. A new construction of the spectrum of classical Virasoro algebra has been given in terms of these ingredients.

Even in view of quantisation, it is worth extending our constructions out of criticality in another way. We may define the anti-chiral transfer matrix

$$\bar{T}(\bar{x}, \lambda) = \mathcal{P} \exp \left( \lambda \int_x^0 dy (e^{-2\bar{\phi}(y)} E + e^{2\bar{\phi}(y)} F) \right) e^{H\bar{\phi}(\bar{x})} \quad (6.1)$$

which solves the linear problem

$$\partial_{\bar{x}} \bar{T}(\bar{x}; \lambda) = \bar{T}(\bar{x}; \lambda) \bar{A}_{\bar{x}}(\bar{x}; \lambda) \quad (6.2)$$

where  $\bar{A}_x(x; \lambda)$  is obtained from  $A_x$  by substitution  $\phi \rightarrow -\bar{\phi}$ . In accordance with [7], we suggest for the off-critical transfer matrix

$$\mathbf{T}(x, \bar{x}; \lambda; \mu) = \bar{T}(\bar{x}; \mu/\lambda)T(x; \lambda), \quad (6.3)$$

and correspondingly for  $\mathbf{Z}(x, \bar{x}; \mu; \lambda)$ .

Following the basic work [7], one quantizes the corresponding mKdV system by replacing the Kac-Moody algebra with the corresponding quantum group and the mKdV field  $\phi$  with the Feigin–Fuchs–Dotsenko–Fateev free field [12]. As explained in [8] the importance of considering also the  $A_2^{(2)}$ -mKdV hierarchies is due to the fact that the quantisation of this second semiclassical system exhausts the integrability directions of theories of type (1.1) starting from Minimal Models of [9]. For different kinds of CFT it is sufficient to consider hierarchies attached to different Kac-Moody algebras (in the Drinfeld-Sokolov scheme [5]).

In conclusion, we have presented a generalization of the dressing symmetry construction leading to a non-local Virasoro symmetry of the mKdV hierarchy and SGM. We stress that it has nothing to do with the space-time Virasoro, generated at the classical level by the moments of the *classical stress tensor*  $\int x^n u(x) dx$ . It is obtained instead by dressing the differential operator  $\lambda^{n+1} \partial_\lambda$ . In view of the relation between the spectral parameter and the on-shell rapidity  $\lambda = e^\theta$ , it is generated probably by diffeomorphisms in the momentum space and in this sense is dual to the space-time Virasoro symmetry. Although we presented a construction only in the case of mKdV, it can be easily extended for the generalized KdV theories as well. Of particular interest is the  $A_2^{(2)}$  hierarchy, connected with the  $\phi_{1,2}$  perturbation of CFT models [8].

Furthermore, such a symmetry appears also in the study of Calogero-Sutherland model whose connection with the matrix models and CFT is well known. Moreover, it is known that in the q-deformed case it becomes a deformed Virasoro algebra [23]. It is natural to suppose that in the same way our construction is deformed off-critically.

We suggest that this Virasoro symmetry could be of great importance for the study of **2D-IQFT**. First of all it should provide a new set of conserved charges closing a non-abelian algebra, thus carrying necessarily more information about the theory. Furthermore one might quantise these charges at conformal and off-critical level. We have reasons to believe that the perturbed version (i.e. SGM) should be closely related to the aforementioned DVA.

Recently, Babelon, Bernard and Smirnov [3] constructed certain null-vectors off-criticality in the context of the form-factor approach. They showed that there is a deep connection, at the classical level, between their construction and the finite zone solutions of KdV and the Witham theory of averaged KdV. On the other hand the Virasoro algebra presented above has a natural action on the finite zone solutions, changing the complex structures of the corresponding hyperelliptic Riemann surfaces, and on the basic objects of the Witham hierarchy [18]. This suggests we may found

the quantum action of our symmetries (in particular the Virasoro one) in SGM using the form factor formalism developed in [3].

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